M317 In Class Exam solutions

1. If $f \in C[0,2]$ then for some finite M, $|f(x)| \le M$ for $0 \le x \le 2$.

   If f is continuous on the compact set $[0,2]$
   then $rng[f]$ is compact by the compact range theorem.
   compact=closed and bounded,
   $rng[f]$ bounded means $|f(x)| \le M$ for $0 \le x \le 2$
   So this is true.

2. If $f \in C[0,1]$ then $rng[f]$ contains all its accumulation points.

   If f is continuous on the compact set $[0,1]$ then
   $rng[f]$ is compact by the compact range theorem.
   compact=closed and bounded,
   closed=contains all its accumulation points
   Then $rng[f]$ contains all its accumulation points so this is true.

3. If $f \in C[a,b]$ and $\{x_n\}$ is convergent, then $\{f(x_n)\}$ is a C-sequence

   If f is continuous on the compact set $[0,1]$ then f is uniformly continuous on $[0,1]$
   If $\{x_n\}$ is convergent, then $\{x_n\}$ is a C-sequence
   Since f is uniformly continuous on $[0,1]$, f maps C-seq's in $[0,1]$ into C-seq's in $rng[f]$
   Then $\{f(x_n)\}$ is a C-sequence so this is true.

4. If $f \in C(a,b)$ then $[f(a)+f(b)]/2$ belongs to $rng[f]$ so ....

   $f(x) = \dfrac{1}{(x-a)(b-x)} \in C(a,b)$ but neither $f(a)$ nor $f(b)$ belong to $rng[f]$
   This is false

5. If $f \in C[0,1]$ and $f(x) < 0$ for all x in $[0,1]$, then for some $A < 0$, $f(x) \le A < 0\ \forall\ x \in [0,1]$.

   If f is continuous on the compact set $[0,1]$ then the extreme value theorem
   asserts that for some x^* in $[0,1]$, $f(x^*) = \sup_I f = A$.
   Since $f(x) < 0$ for all x in $[0,1]$, $A < 0$
   Then $f(x) \le f(x^*) = \sup_I f = A < 0$ for all x in $[0,1]$ so this is true

6. If $\{a_n\} \subset dom[f]$ converges to A and $\{f(a_n)\}$ converges to $f(A)$, then f is continuous at $x = A$.

   this is false (if, for ALL sequences $\{a_n\} \subset dom[f]$ converging to A
   we had $f(a_n) \to f(A)$ it would be true.)
   Consider

$$f(x) = \begin{cases} \frac{x}{|x|} & if \quad x \neq 0 \\ 1 & if \quad x = 0 \end{cases}$$

Then $x_n = \frac{1}{n} \to 0$ and $f(x_n) = 1 \to 1 = f(0)$ but f is not continuous at $x = 0$.

7. If $f \in C(0,1)$ and $f(x_0) > 0$ for some x_0 in $(0,1)$ then $f(x) > 0$ for all $x \in N_\varepsilon(x_0)$ for some $\varepsilon > 0$.

Since $f(x_0) > 0,\ f(x_0)/10 > 0$ and for this positive number there exists $\varepsilon > 0$ such that

$$|f(x) - f(x_0)| < \frac{f(x_0)}{10} \quad \textit{for all } |x - x_0| < \varepsilon$$

$$-\frac{f(x_0)}{10} < f(x) - f(x_0) < \frac{f(x_0)}{10} \quad \textit{for all } |x - x_0| < \varepsilon$$

$$f(x_0) - \frac{f(x_0)}{10} < f(x) < f(x_0) + \frac{f(x_0)}{10} \quad \textit{for all } |x - x_0| < \varepsilon$$

Since $f(x_0) - \frac{f(x_0)}{10} > 0,$ we have $f(x) > 0$ for all $x \in N_\varepsilon(x_0)$ so this is true.

8. If $f \in C[0,1]$ and $x < y$ implies $f(x) < f(y)$ for all x,y in $[0,1]$ then f has a continuous inverse.

If $f \in C[0,1]$ and $x < y$ implies $f(x) < f(y)$ for all x,y in $[0,1],$ then f is continuous and strictly increasing. This is sufficient to imply (by the continuous inverse theorem) that f^{-1} exists and is continuous.